Technical Notes

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On the Not-So-Slender Wing Theory

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Introduction

TO obtain the near-field solution of a slender body, or a slender wing, in a compressible flow, the usual practice is to make use of the integral transform method. This method has been extensively employed by Ward,[1] Adams-Sears,[2] Miles,[3] and Landahl,[4] among others, in obtaining the steady of unsteady slender body/wing potentials. Although the method is considered to be most useful, the inversion transform of it in some cases could become rather involved. When the solution is derived in the physical plane, the direct procedures based on various series expansion methods in the past appear to be too laborious (e.g., Refs. 5 and 6).

In this Note, we present a much simpler method of obtaining the 'not-so-slender' body potentials (denoted by NSSB hereafter, or equivalently the 'not-so-slender' wing potentials, see Ref. 2) and their antisymmetic (lifting) potentials directly in the physical plane. The present method is based on an uncovered asymptotic property of a line source structure near the body axis, with the slender-body solution being the lowest order term [see Eqs. (3)]. It can be shown that the subsequent higher-order terms involve no newer functional relation than that already found in the slender-body solution. General recurrence formulas may then be derived for the particular NSSB potentials in the linearized field of subsonic, supersonic, and sonic flows.

The Recurrence Formulas

Let the linearized perturbation potential be $\varphi(x,r)$; x and r are the axial and radial coordinates. Then the linearized axisymmetric potential equation can be expressed in the general form as

$$\varphi_{rr} + (1/r)\varphi_r = \Lambda\varphi \tag{1a}$$

where

$$\Lambda = -\beta^2(\partial^2/\partial x^2) \tag{1b}$$

for subsonic flow ($\beta^2 = 1 - M_\infty^2$)

$$\Lambda = B^2(\partial^2/\partial x^2) \tag{1c}$$

for supersonic flow ($B^2 = M_\infty^2 - 1$), and M_∞ is the freestream Mach number. For the case of sonic flow,

$$\Lambda = \Gamma(\partial/\partial x) \tag{1d}$$

based on the parabolic equation proposed by Oswatitsch and Keune[7]; or

$$\Lambda = a^2(x - x^*)(\partial^2/\partial x^2)$$

based on the approximate equation proposed by Cole and Royce.[8] The constants Γ, a^2 and the sonic point location x^* can be uniquely determined according to Hosokawa's condition (e.g., see Ref. 6). We now express the solution of Eq. (1) in the asymptotic form for small r

$$\varphi(x,r) \sim \varphi^{(0)}(x,r) + \Delta\varphi^{(1)}(x,r) + \ldots \tag{2}$$

where $\varphi^{(0)}$ is the slender body potential and $\Delta\varphi^{(1)}$ is the NSSB potential. The potential $\varphi^{(0)}$ can be written in the form of

$$\varphi^{(0)}(x,r) = f(x)\ln r + g(x) \tag{3a}$$

and the NSSB potential (correction) sought will be in the form of

$$\Delta\varphi^{(1)}(x,r) = h(x)r^2\ln n\; r + k(x)r^2 \tag{3b}$$

The line source distribution function $f(x)$, is related to the body cross-sectional area variation from the slender-body tangency condition, whereas the spatial influence function $g(x)$, generally expressed as a composite function of $f(x)$, must be obtained from the total solution in satisfying the far-field conditions. Since the axisymmetric slender body potential is assumed given, various $g(x)$'s can be listed in different linearized flow regimes as follows

1) Subsonic flow, $M_\infty < 1$ (e.g., see Ref. 6)

$$g = g_\beta(x) = f(x)\ln\beta - \frac{1}{2}\frac{\partial}{\partial x}\int_0^x f(\xi)\ln[2(x-\xi)]\,d\xi + \frac{1}{2}\frac{\partial}{\partial x}\int_x^1 f(\xi)\ln(2(\xi - x)]\,d\xi \tag{4a}$$

2) Supersonic flow, $M_\infty > 1$ (e.g., see Ref. 5)

$$g = g_B(x) = f(x)\ln B - \frac{\partial}{\partial x}\int_0^x f(\xi)\ln[2(x-\xi)]\,d\xi \tag{4b}$$

3) Linearized model of sonic (or transonic) flow, $M_\infty = 1.0$ (e.g. see Ref. 7).

$$g = g_\Gamma(x) = f(x)\ln\left(\frac{\Gamma e^c}{4}\right)^{1/2} - \frac{1}{2}\frac{\partial}{\partial x}\int_0^x f(\xi)\ln(x-\xi)\,d\xi \tag{4c}$$

where $c = 0.5772...$, Euler's constant. It should be noted that for the case of slender wing, the first term of Eq. (3a), according to the area rule, is to be replaced by a spanwise integral term, i.e.

$$\frac{1}{\pi}\frac{\partial}{\partial x}\int_{-S(x)}^{S(x)} \bar{f}(x,\eta)\ln|y-\eta|\,d\eta$$

where $\bar{f}(x,\eta)$ is the surface-source distribution function.

Next, we turn to the asymptotic expansion for small r admissible to Eq. (1). It is essential to point out that $\Delta\varphi^{(1)}$, etc. turn out to involve no other functional relation than that between $g(x)$ and $f(x)$. The result can be readily generated as follows. Assume

$$\varphi(x,r) = \varphi^{(0)}(x,r) + \sum_{n=0}^{N}\Delta\varphi^{(n+1)} + O(r^{2(N+2)}\ln r) \tag{5a}$$

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and

$$\Delta\varphi^{(n+1)} = \varphi^{(n+1)} - \varphi^{(n)} \sim 0(r^{2(n+1)}\ln r) \tag{5b}$$

where N is the finite number terms taken. If we substitute Eq. (5) into Eq. (1) and collect terms of like order, the following equations result

$$\varphi_{rr}^{(0)} + (1/r)\,\varphi_r^{(0)} = 0 \tag{6a}$$

$$\varphi_{rr}^{(n+1)} + (1/r)\varphi_r^{(n+1)} = \Lambda\varphi^{(n)} + 0(r^{2(n+1)}\ln r) \tag{6b}$$

($n=0, 1, 2...$). Clearly, Eq. (3a) is a solution of Eq. (6a). For the higher-order solutions, Eq. (6b) can be integrated immediately resulting in a general recurrence formula

$$\varphi^{(n+1)}(x,r) = \varphi_c(x,r) + \Lambda\int^r\int^\eta \frac{1}{\eta}\rho\varphi^{(n)}(x,\rho)\,d\rho\,d\eta \tag{7}$$

where $\varphi_c(x,r) = C_1(x)\ln r + C_2(x)$ is yet an unknown function. However, $\varphi_c(x,r)$ is identified to be $\varphi^{(0)}(x,r)$, as suggested by the asymptotic development sought in Eqs. (2) and (3a). Thus, $C_1(x)$ and $C_2(x)$ are no more than $f(x)$ and $g(x)$, respectively.

For the case of $n=0$, we obtained the NSSB potential in a general compact formula

$$\Delta\varphi^{(1)} = (r^2/4)\Lambda[f(x)(\ln nr - 1) + g(x)] \tag{8a}$$

For corresponding expression of Λ [Eqs. (1b–e)] and $g(x)$ [Eqs. (4a-c)] Eq. (8a) checks identically with the previous solutions for subsonic, supersonic, and sonic flows (e.g., see Refs. 2, 5, and 6).

The linearized antisymmetric (lifting) potential $\psi(x,r)$ can be generally expressed in terms of the partial derivative of the axisymmetric potential as (e.g., see Eq. (2a) Ref. 14)

$$\hat{\varphi}(x,r,\theta) = A\psi(x,r)\cos\theta = A(\partial\varphi/\partial r)\cos\theta \tag{9a}$$

where $\hat{\varphi}(x,r,\theta)$ is the cross flow potential, and θ is the azimuthal angle, and $A = 1/2\pi$.

Clearly, $\psi(x,r)$ satisfies the following equation, i.e.

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\psi)\right] = \Lambda\psi \tag{9b}$$

Hence, similar successive approximation scheme follows for the antisymmetric potential $\psi(x,r)$. Consequently, we obtain the recurrence formula for ψ, i.e.

$$\psi^{(0)}(x,r) = [F(x)/r] \qquad \text{(Munk-Jones Potential)} \tag{10a}$$

$$\psi^{(n+1)}(x,r) = \psi^{(0)}(x,r) + \frac{\Lambda}{r}\int^r \rho\varphi^{(n)}(x,\rho)\,d\rho \tag{10b}$$

For the case of $n=0$, the general compact formula for $\Delta\psi^{(1)}$ reads

$$\Delta\psi^{(1)}(x,r) = (r/2)\Lambda[F(x)(\ln r - 1/2) + G(x)] \tag{11}$$

where $F(x)$ is the dipole distribution function, again to be determined by the tangency condition and by the Adams-Sears iterative procedure. The function $G(x)$ adopts simply the previous expression of $g(x)$, Eqs. (4), except that in which $f(x)$ is replaced by $F(x)$. Similarly, the total antisymmetric potential can be written as

$$\hat{\varphi}(x,r,\theta) = A\left\{\psi^{(0)}(x,r) + \sum_{n=0}^{N}\Delta\psi^{(n+1)} + 0(r^{2N+3}\ln r)\right\}\cos\theta \tag{12a}$$

$$\Delta\psi^{(n+1)} = \psi^{(n+1)} - \psi^{(n)} \sim 0(r^{2n+1}\ln nr) \tag{12b}$$

Eq. (11) also checks identically with previous solutions for all flow regimes.

It is noted that solutions (10a) and (11) were employed as standard solutions in working out the stability derivatives for slender bodies (see Refs. 2, 5, and 6).

New Sonic-Flow Solutions

Cole and Royce[8] approximated the sonic equation with $\Lambda = a^2(x-x^*)\,(\partial^2/\partial x^2)$ and assumed $f'(x^*)=0$; they have derived the slender-body axisymmetric solution as

$$g = g_a(x) = f(x)\ln\frac{a}{2} + \frac{1}{2}\int_0^x f'(\xi)\ln\xi\,d\xi$$
$$-\frac{1}{2}\left\{\int_0^x f'(\xi)\ln(x-\xi)\,d\xi + \int_{x^*}^x f'(\xi)\ln|x-\xi|\,d\xi\right\} \tag{13}$$

To demonstrate the present method, we can simply write down the antisymmetric solution readily from the formulas (11) and (12), using the functional relation provided by Eq. (13) in the same manner as the application of Eq. (4) in conjunction with Eq. (11)., i.e.

$$\hat{\varphi}(x,r,\theta) = (\cos\theta/2\pi)[F(x)/r + \Delta\psi^{(1)}]$$

$$\Delta\psi^{(1)} = \frac{1}{4}a^2r(x-x^*)\left\{F'(x)\left[\ln\frac{a^2r^2}{4} - 1\right]\right.$$
$$+\frac{\partial^2}{\partial x^2}\int_0^x F'(\xi)\ln|\xi|\,d\xi - \frac{\partial^2}{\partial x^2}\int_0^x F'(\xi)\ln(x-\xi)\,d\xi$$
$$\left.-\frac{\partial^2}{\partial x^2}\int_{x^*}^x F'(\xi)\ln|x-\xi|\,d\xi\right\}$$

For the sonic slender-wing solution, a counterpart of the slender-body solution, the nonlifting case has been given by Burg.[9] The lifting case consists of two parts, namely

$$\hat{\varphi}_z(x,y,0) = -\frac{1}{\pi}\int_{-S(x)}^{S(x)}\frac{\hat{\varphi}_\eta\,d\eta}{y-\eta} + \zeta[\hat{\varphi};a^2] + 0(a^4\sigma^4\ln a\sigma) \tag{15a}$$

where $S(x)$ is the half span, and σ is the semi-span-to-chord ratio.

The first term of Eq. (15a) is the Munk-Jones term, corresponding to the first term of Eq. (14). The second term of Eq. (15a) can be conveniently obtained from the second term of Eq. (14) in replacing $r\cos\theta$ by z, r by $\sqrt{(y-\eta)^2+z^2}$ and $F(x)$ by $\int_{-S(x)}^{S(x)}\hat{\varphi}(x,\eta,0)\,d\eta$, then differentiating it with respect to z and letting z approach zero, i.e.,

$$\zeta[\hat{\varphi};a^2] = \frac{a^2}{4}(x-x^*)\left[\frac{\partial}{\partial x}\left\{\left[\ln\frac{a^2}{4}+1\right]\int_{-S(x)}^{S(x)}\hat{\varphi}\,d\eta\right.\right.$$
$$\left.+2\int_{-S(x)}^{S(x)}\hat{\varphi}\ln|y-\eta|\,d\eta\right\}$$
$$+\frac{\partial^2}{\partial x^2}\left\{\int_0^x[\hat{\varphi}_s+\hat{\varphi}_{-s}]S'(\xi)\ln\frac{|\xi|}{(x-\xi)}\,d\xi\right.$$
$$\left.\left.-\int_{x^*}^x[\hat{\varphi}_s+\hat{\varphi}_{-s}]S'(\xi)\ln|x-\xi|\,d\xi\right\}\right] \tag{15b}$$

where $\hat{\varphi}=\hat{\varphi}(x,\eta,0)$, $\hat{\varphi}=\hat{\varphi}(\xi, S(\xi), 0)$ and $\hat{\varphi}_{-s}=\hat{\varphi}(\xi,-S(\xi),0)$

The above solution is derived only for the configuration of slender pointed wings. The value of a^2 can be determined through the equivalent body of revolution according to the area rule (Ref. 9). A similar concept was suggested by the author in Ref. 10 for unsteady sonic flow calculation based on the parabolic method. Thus, for calculation of the steady force and moment for bodies and wings, we employ the

Adams-Sears iterative procedure, which is straightforward once the solutions (14) and (15) are known (e.g., see Ref. 2 and Ref. 4, pp. 34-40)

We remark that the solutions (14) and (15) appear to be new in the literature, which can be otherwise obtained from more complicated methods such as integral transform. Furthermore, we note that these solutions are strictly near-field solutions, derived for bodies/wings of slender ratio upon a very small departure from the nonlifting flow (hence, a thickness dominated problem). They should be therefore distinguished from the recent work of Cheng and Hafez[11] in which three-dimensionality is considered for the nonlinear transonic flowfield involving different degrees of lift control.

Conclusions

A much simpler derivation for the not-so-slender wing/body potential has been presented as a result of the uncovered asymptotic property of the line source structure near the body axis. Based on the asymptotic equations formulation, a general recurrence formula is obtained which can yield readily the NSSB potentials up to arbitrary order in r. Formulas Eq. (8a) and Eq. (11) are generally valid for a smooth body in all regimes of the potential flow insofar as the governing equations are linear. Also, the linear operator Λ is restricted to performing differentiation only in the x-direction.

The present approach can be readily extended to the unsteady flow analysis. To obtain the oscillatory NSSB potential, however, the spatial influence function, $g(x)$ [see Eq. (3a)], corresponding to the axisymmetric pulsating solutions, must be provided by other means of derivation. In fact, these pulsating solutions have been previously established by Platzer[12] for subsonic and supersonic cases and generalized by Liu, et al.,[6] [Eqs. (2.17)-(2.20), p. 3] to include the linearized transonic flow case. It is demonstrated in Ref. 13 that following the present approach the oscillatory NSSB potential for the linearized transonic flow can be derived within a few steps. Further work concerning the unsteady-flow extension will be reported in a forthcoming Note.

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Efficiency of Navier-Stokes Solvers

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THIS paper describes numerical experiments used to evaluate the relative efficiency of some finite-difference methods for the solution of the vorticity-stream function form of the two-dimensional incompressible Navier-Stokes equations. The comparisons were drawn by recording the CPU time required to obtain a solution as well as the accuracy of this solution using five numerical methods: central differences, first-order upwind differences, second-order upwind differences, exponential differences (these four methods incorporate the Gauss-Seidel iterative procedure), and an ADI solution of the central difference equations. The mesh sizes used were 11×11, 21×21, and 31×31.

Solutions were obtained for two test cases: a recirculating eddy inside a square cavity with a moving top, and an impinging jet flow. Solutions for various Reynolds numbers were obtained, with emphasis on high Reynolds number flows.

Equations and Test Problems

The vorticity-stream function formulation of the two-dimensional Navier-Stokes equations is given by the vorticity equation

$$\frac{\partial}{\partial x}(u\zeta)+\frac{\partial}{\partial y}(v\zeta)=\nu\left[\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right] \tag{1}$$

and the stream function equation

$$-\zeta=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} \tag{2}$$

where ζ is the vorticity, ψ is the stream function, u, v, x, y are the velocity components and spatial coordinates, respectively, and ν is the kinematic viscosity. The velocity components are

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